So, what exactly is a qualitative calculus?

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Abstract

The paradigm of algebraic constraint-based reasoning, embodied in the notion of a qualitative calculus, is studied within two alternative frameworks. One framework defines a qualitative calculus as "a non-associative relation algebra (NA) with a qualitative representation", the other as "an algebra generated by jointly exhaustive and pairwise disjoint (JEPD) relations". These frameworks provide complementary perspectives: the first is intensional (axiom-based), whereas the second one is extensional (based on semantic structures). However, each definition admits calculi that lie beyond the scope of the other. Thus, a qualitatively representable NA may be incomplete or non-atomic, whereas an algebra generated by JEPD relations may have non-involutive converse and no identity element. The divergence of definitions creates a confusion around the notion of a qualitative calculus and makes the "what" question posed by Ligozat and Renz actual once again. Here we define the *relation-type qualitative calculus* unifying the intensional and extensional approaches. By introducing the notions of weak identity, inference completeness and Q-homomorphism, we give equivalent definitions of qualitative calculi both intensionally and extensionally. We show that "algebras generated by JEPD relations" and "qualitatively representable NAs" are embedded into the class of relation-type qualitative algebras.

1. Introduction

Qualitative calculi are algebraic languages used for qualitative constraint-based reasoning over relational knowledge. The "qualitative calculus" paradigm emerged and has been shaped in the Qualitative Spatial and Temporal Reasoning field (Aiello et al., 2007; Ligozat, 2013). A recent survey (Dylla et al., 2017) outlines over 40 families of qualitative spatio-temporal calculi, each covering one particular aspect of time or space, e.g. topology, direction, orientation, relative duration, etc. Among standard examples are Allen's interval calculus (Allen, 1983), Region Connection Calculi RCC-5 and RCC-8 (Randell et al., 1992), and Oriented Point Relation Algebras $\mathcal{OPRA}_m$ (Moratz, 2006). However, the paradigm of algebraic constraint-based reasoning has relevance within a broader scope of knowledge representation and reasoning. For example, there are qualitative calculi developed in the field of ontology matching (Inants and Euzenat, 2015; Inants et al., 2016). Moreover, there are algebraic calculi that combine both qualitative and quantitative information (Kautz and Ladkin, 1991; Hirsch, 1996; Meiri, 1996; Ouaknine and Worrell, 2008; Li and Liu, 2010; Inants et al., 2016).

Frameworks that study qualitative calculi have evolved in two different directions. One approach is representation-based: it started by defining qualitative calculi as relation-algebraic representations (Ladkin and Maddux, 1994), then the definition was weakened to "weak representations" (Ligozat and Renz, 2004), then strengthened to "semi-strong representations" (Mossakowski et al., 2006), and finally expanded to "qualitative representations" (Hirsch et al., 2019). We call this approach *intensional*, since it deals with

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abstract relations and describes their properties axiomatically. In contrast, the second approach deals with sets of concrete binary relations and defines operations on them. It started with defining *weak composition* on strong partition schemes (Ligozat and Renz, 2004), then was generalized to arbitrary jointly exhaustive and pairwise disjoint (JEPD) relations (Dylla et al., 2013) by introducing *weak converse*. We call this approach *extensional*, since it is based on concrete binary relations.

The problem is that the two definitions diverge: each of them admits calculi that lie beyond the scope of the other. The algebras that correspond to each definition differ in their properties, as shown in Table 1. For instance, the algebra CDR generated by cardinal direction relations (Skiadopoulos and Koubarakis, 2004) is not a non-associative relation algebra, as it is required by the intensional definition. On the other hand, the infinite algebra of qualitative distances on a real line (Hirsch et al., 2019) is not complete, thus cannot be described as an algebra generated by JEPD relations.

Table 1: Qualitatively representable algebras (Hirsch et al., 2019) in comparison with algebras generated by JEPD relations (Dylla et al., 2013).

Algebraic property	QRA algebras	QAJEPD algebras
converse is involutive ($r^{\smile\smile} = r$)	yes	not necessarily
distinguished identity element	yes	not necessarily
completeness	not necessarily	yes
atomicity	not necessarily	yes

To overcome this divergence, we introduce the notions of Q-representation, inference completeness and weak identity, enabling to expand the former intensional and extensional definitions of qualitative calculi. We prove that the resulting definitions of qualitative calculi are equivalent and show that they define the broadest class of algebras with respect to a predefined set of characteristic properties. Finally, we embed the known classes of qualitative calculi into this general framework.

Qualitative calculi of arity greater than two, as well as non-relation-type calculi, lie beyond the scope of this paper.

The rest of the paper is structured as follows. Section 2 provides the necessary algebraic background. In Section 3, we give some historical notes and discuss the state-of-the-art definitions of qualitative calculi. We specify the class of qualitative algebras corresponding to each definition. Section 4 gives a general perspective on the algebraic approach to constraint-based reasoning and defines the notion of a relation-type qualitative calculus within this context. Section 5 defines the class of relation-type qualitative algebras in two equivalent ways: intensional and extensional. In Section 6, we embed “qualitatively representable NAs” and “algebras generated by JEPD relations” into the class of qualitative algebras defined in Section 5. We provide intensional and extensional classification of important subclasses of qualitative algebras. Finally, Section 7 gives concluding remarks and directions for future work.

2. Preliminaries

We assume familiarity with basic universal-algebraic notions such as signature, algebra, subalgebra, homomorphism, isomorphism, direct product, etc. (Burris and Sankappanavar, 1981).

An algebra $\mathcal{B} = (B, +, ^{-})$ with a binary operation $+$ called *Boolean sum* and a unary operation $^{-}$ called *complement* is called a *Boolean algebra* if it satisfies the following equations:

$$a + b = b + a, \qquad (a + b) + c = a + (b + c), \qquad \overline{(\overline{a} + \overline{b})} + \overline{(\overline{a} + b)} = a. \tag{1}$$

The standard derived operations and constants are: *Boolean product* $a \cdot b = \overline{(\overline{a} + \overline{b})}$, *unit* element $1 = a + \overline{a}$, *zero* element $0 = \overline{1}$. An example of a Boolean algebra is the powerset of any set X, denoted by $\wp(X)$, together with set union $\cup$ and complement $-_X$ defined as $-_X(Y) = X \backslash Y$. Subalgebras of powerset Boolean algebras $\wp(X)$ are called *fields of sets*. Every Boolean algebra is isomorphic to a field of sets. A partial order $\leq$ is defined on B as $a \leq b$ iff $a + b = b$. A non-zero element a of a Boolean algebra is called an *atom* if $0 \leq b \leq a$ implies that $b = 0$ or $b = a$ for any b. A Boolean algebra is *atomic* if every element is a supremum of atoms below it (Birkhoff, 1973). The set of all atoms of $\mathcal{B}$ is denoted by $At(\mathcal{B})$. For any element x of $\mathcal{B}$, $At(x)$ denotes the set of those atoms a, for which $a \leq x$.

A Boolean algebra $\mathcal{B}$ is said to be *complete*, if the poset $(B, \leq)$ has all suprema and all infima. Every complete atomic Boolean algebra is isomorphic to a powerset Boolean algebra. If $\mathcal{X}$ is a set of sets, then we will use a shorter notation $\cup\mathcal{X}$ instead of $\cup_{X \in \mathcal{X}} X$, and $\cap\mathcal{X}$ for $\cap_{X \in \mathcal{X}} X$.

An expansion of a Boolean algebra with one or more operations is called a *Boolean algebra with operators* (BAO) if each of these operations is additive in every argument (Jónsson and Tarski, 1951). A BAO is atomic if so is its Boolean reduct. A BAO is said to be complete if its Boolean reduct is complete and if each of its operators is completely additive.

A *relation-type algebra* is an algebra with a signature consisting of Boolean operations $+, \cdot, ^-, 0, 1$, a binary operation '$;$' called *relative product*, a unary operation '$^\smile$' called *converse*, and a constant '$1'$' called the *identity* element. A relation-type algebra $\mathcal{A} = (A, +, \cdot, ^-, 0, 1, ;, ^\smile, 1')$ is called a *non-associative relation algebra*[1] (NA) (Maddux, 1982), if

1) the reduct $(A, +, \cdot, ^-, 0, 1)$ is a Boolean algebra,

2) $1'; x = x; 1' = x$ (identity law),

3) $(x; y) \cdot z = 0 \Leftrightarrow (x^\smile; z) \cdot y = 0 \Leftrightarrow x \cdot (z; y^\smile) = 0$ (Peircean law),

for all $x, y, z \in A$. A non-associative relation algebra $\mathcal{A}$ is called

a) a *weakly-associative relation algebra*, if $(1' \cdot x); (1; 1) = ((1' \cdot x); 1); 1$,

b) a *semi-associative relation algebra*, if $x; (1; 1) = (x; 1); 1$,

c) a *relation algebra*, if $(x; y); z = x; (y; z)$,

for all $x, y, z \in A$. The respective classes of algebras are denoted by NA, WA, SA and RA. The following inclusions hold: $\mathsf{NA} \supset \mathsf{WA} \supset \mathsf{SA} \supset \mathsf{RA}$.

Fact 1 (Maddux (1982))**.** *In non-associative relation algebras relative composition and converse are completely additive.*

The set $\wp(U \times U)$ of binary relations over U, together with Boolean set operations, composition $\circ$, converse $^{-1}$ and identity Id_U is a relation algebra, denoted by $\mathfrak{Re}(U)$.

Any complete atomic (particularly any finite) non-associative relation algebra $\mathcal{A}$ is fully specified by its *atom structure*. An atom structure consists of the set of atoms $At(\mathcal{A})$, the set of identity atoms $At(1') \subseteq At(\mathcal{A})$, the converse restricted to atoms $^\smile : At(\mathcal{A}) \to At(\mathcal{A})$ and the *composition table*. A composition table is a function $CT : At(\mathcal{A}) \times At(\mathcal{A}) \to \wp(At(\mathcal{A}))$, defined by $z \in CT(x, y)$ iff $(x; y) \cdot z \neq 0$. The triples (x, y, z), where x, y and z are atoms and which satisfy $(x; y) \cdot z \neq 0$ are called *consistent triples*.

3. State of the art: intensional and extensional definitions of qualitative calculi

In a nutshell, a qualitative calculus consists of two components: one symbolic and one semantic. The symbolic component is an abstract algebra of relations, and the semantic component is a function that interprets the relation symbols as relations over some set. The intensional approach to defining a qualitative calculus starts with an abstract algebra that satisfies certain axioms, whereas the extensional approach starts with a semantic structure, namely a system of JEPD relations.

[1]The name "non-associative relation algebra" is somewhat confusing, as it is meant to denote a "not necessarily associative relation algebra".

3.1. Intensional approach

The intensional approach dates back to the paper of Ladkin and Maddux (1994), in which the paradigm of algebraic constraint-based reasoning introduced by Allen (1983) was generalized to relation algebras with a representation. Such a framework assumes a perfect match between the symbolic (intensional) and the actual (extensional) composition of relations. Then Ligozat and Renz (2004) defined a qualitative calculus as "a non-associative relation algebra with a weak representation". The notion of a weak representation allows symbolic composition to be an upper approximation of the relational composition. Not only does "weak representation" describe the relation between abstract and actual composition, but also allows to cast the consistency problem as two weak representations' compatibility.

However, the definition of Ligozat and Renz was criticized for several shortcomings. First, weak representation is overly permissive when it comes to describing qualitative calculi, since it allows an unnecessary loss of information. It was suggested to narrow the framework down to "best upper approximations" of composition, which led to the notion of a semi-strong representation (Mossakowski et al., 2006). The second shortcoming is that weak representations are not required to be injective, which is the case in all qualitative calculi (Dylla et al., 2013). Indeed, a non-injective map corresponds to the case when some relation symbols are interpreted as the empty relation. Finally, it was pointed out that weak representations allow non-atomic identity elements (Inants, 2016), which means that partition schemes in the sense of Ligozat and Renz (2004) are not the only semantic structures that correspond to weak representations.

The definition of a qualitative calculus proposed by Hirsch et al. (2019) overcomes the abovementioned shortcomings and generalizes that of Ligozat and Renz (2004) to the infinite case. This generalization is motivated by several efforts to combine qualitative calculi with quantitative information, which results in infinitely many relations. An interesting feature of this definition is that it does not require qualitative calculi to be based on JEPD relations.

Summing up, the state-of-the-art definition of a qualitative calculus within the representation-based intensional approach is that "qualitative calculus is a non-associative relation algebra with a qualitative representation".

Definition 1 (Hirsch et al. (2019))**.** Let $\mathcal{A} = (A, +, \cdot, ^-, 0, 1, ;, ^\smile, 1')$ be a non-associative relation algebra. A *qualitative representation* of the algebra $\mathcal{A}$ over a set U is a function $\varphi : A \to \wp(U \times U)$, such that

1. φ is an embedding of Boolean algebras:
   (a) φ is injective
   (b) $\varphi(a + b) = \varphi(a) \cup \varphi(b)$
   (c) $\varphi(a^-) = (U \times U) \setminus \varphi(a)$
2. $\varphi(1') = Id_D$
3. $\varphi(a^\smile) = (\varphi(a))^{-1}$,
4. $a; b \leq c \leftrightarrow \varphi(a) \circ \varphi(b) \subseteq \varphi(c)$

for all $a, b, c \in A$.

A qualitative representation is a *(strong) representation* iff $\varphi(a; b) = \varphi(a) \circ \varphi(b)$ for all $a, b \in A$. If $\mathcal{A}$ has a qualitative representation, then it is called a *qualitatively representable algebra*. QRA denotes the class of all qualitatively representable algebras.

Example 1 (Hirsch et al. (2019))**.** This is an example of an infinite relation algebra for expressing metric constraints on a linearly ordered metric space. Its elements are finite unions of real intervals, e.g. $(2, 5) \cup [6, 8]$. There is one identity atom, namely $[0, 0]$, converse is defined by $(m, n)^\smile = (-n, -m)$, and composition is defined by $(m, n); (m', n') = (m + m', n + n')$ for $m > n$, $m' > n'$, with similar definitions for closed and semi-open intervals. A strong representation φ over the real numbers may be obtained by letting $(x, y) \in \varphi((m, n)) \Leftrightarrow m < y - x < n$, with similar definitions for closed and semi-open intervals. This provides a useful way of expressing metric constraints between points, e.g. the constraint $(x, y) \in \varphi([-3, -2] \cup [2, 3])$ means that the distance between x and y is at least two and not more than three.

3.2. Extensional approach

The extensional approach starts with a set of jointly exhaustive and pairwise disjoint relations (a partition of $U \times U$). Any finite set $\mathcal{P}$ of nonempty JEPD relations on a set U generates a *concrete qualitative algebra*

$$\mathcal{Q}^*(\mathcal{P}) = (\mathcal{P}_\cup, \cup, \cap, -_{U\times U}, \varnothing, U\times U, \diamond, \breve{\ }), \tag{2}$$

where $\mathcal{P}_\cup$ is the union closure of $\mathcal{P}$ ($\mathcal{P}_\cup = \{\cup X \mid X \subseteq \mathcal{P}\}$), $\diamond$ is called *weak composition* Ligozat and Renz (2004) and $\breve{\ }$ is called *weak converse* (Dylla et al., 2013).

$$R \diamond S = \cap\{T \in \mathcal{P}_\cup \mid T \supseteq R \circ S\}, \qquad R^{\smile} = \cap\{T \in \mathcal{P}_\cup \mid T \supseteq R^{-1}\}. \tag{3}$$

Operations $\diamond$ and $\breve{\ }$ are additive. A qualitative calculus in the sense of Dylla et al. (2013) is a finite Boolean algebra with operators

$$\mathcal{A} = (\wp(Rel), \cup, \cap, -_{Rel}, \varnothing, Rel, ;, \breve{\ }), \tag{4}$$

where Rel is a set of relation symbols, together with a partition scheme $\mathcal{P}$ and a function $\varphi : \wp(Rel) \to \mathcal{P}_\cup$, such that

1. φ is an isomorphism between Boolean algebras $(\mathcal{P}_\cup, \cup, \cap, -_{U\times U}, \varnothing, U \times U)$ and $(\wp(Rel), \cup, \cap, -_{Rel}, \varnothing, Rel)$,
2. $\varphi(a;b) \supseteq \varphi(a) \diamond \varphi(b)$
3. $\varphi(a^{\smile}) \supseteq \varphi(a)^{\smile}$,

for all $a, b \in A$. If $\varphi(a;b) \supset \varphi(a) \diamond \varphi(b)$, then the relative product (;) is weaker than weak composition ($\diamond$), similarly for converse. As pointed out in (Mossakowski et al., 2006), "weaker than weak" composition allows an unnecessary loss of information. In addition, (Hirsch et al., 2019) shows several shortcoming of *feeble* representations – weaker variants of qualitative representations that correspond to "weaker than weak" composition. In particular, a feeble representation of an algebra $\mathcal{A}$ does not represent $\mathcal{A}$, because some consistent triples may be absent in the representation. Due to these shortcomings, we disregard "weaker than weak" operations in this paper. If a qualitative calculus has at least weak composition and converse, then the interpretation function φ is an isomorphism between $\mathcal{A}$ and $\mathcal{Q}^*(\mathcal{P})$. Thus, the class of such qualitative algebras is defined as the class of isomorphic copies of algebras $\mathcal{Q}^*(\mathcal{P})$, denoted by QAJEPD. Note that QAJEPD algebras do not have the same signature as QRA, since the former have no distinguished identity elements.

Example 2. The language of cardinal direction relations (CDR) is introduced in (Skiadopoulos and Koubarakis, 2004). The universe is the set of regions of the Euclidean plane $\mathbb{R}^2$ with a coordinate system. The relation symbols are 3×3 binary matrices. A relation symbol $r = (r_{ij})$, where $i, j \in \{1, 2, 3\}$, corresponds to a binary relation R defined as follows. A pair of regions (A, B) belongs to R iff the intersection of A with (i, j)-th partition of space created by the bounding rectangle of B is empty if and only if $r_{ij} = 0$.

A less general definition that follows the extensional approach is given by Westphal et al. (2014). Qualitative algebras are defined as isomorphic copies of algebras generated by strong partition schemes. Such algebras have an identity element, hence are relation-type algebras. Moreover, they are weakly-associative relation algebras, hence can be specified compactly by a structure called a "notion of consistency" (Hodkinson, 1997).

3.3. Discussion

The intensional and extensional approaches provide complementary perspectives. However, they have different scopes. For instance, the algebra of the calculus in Example 1 is not complete, hence does not belong to QAJEPD. On the other hand, the algebra generated by the partition scheme of Example 2 does not satisfy the property $r^{\smile\smile} = r$ and neither has an identity element, hence it does not belong to QRA. In the rest of the paper, we will embed QAJEPD and QRA into a larger class of qualitative algebras, which will be given two equivalent definitions: intensional and extensional.

4. Contextual definition of a qualitative calculus

Having alternative definitions of the same object is very common in mathematics. For instance, the definition of a Boolean lattice and that of a Boolean algebra are equivalent, since they correspond to the same class of mathematical objects. The situation with qualitative calculi is different. There are alternative definitions of qualitative calculi not equivalent to each other, and moreover, there is no definition more general than the others. Why is it so? The fact that various definitions assign the same name to the defined objects suggests that there is a certain implicit property shared by all qualitative calculi. This property, however, has never been made explicit, which resulted in perpetual expansion of the class of objects called qualitative calculi. In this section, we formalize the property that turns an abstract algebra of relation symbols into a qualitative calculus, thus giving an "external", contextual definition thereof.

4.1. Constraint language

Here we specify a logic $\mathcal{CL}$ for expressing relational constraints. We distinguish between simple constraints $c(x_1, x_2)$ and complex constraints C defined inductively as follows:

$$c(x_1, x_2) ::= r(x_1, x_2) \mid c_1(x_1, x_2) \vee c_2(x_1, x_2) \mid \neg c(x_1, x_2), \tag{5}$$

$$C ::= c(x_1, x_2) \mid C_1 \wedge C_2, \tag{6}$$

where r is a binary relation symbol and x_1, x_2 are variables. Notice that disjunction is allowed only between constraints over the same pair of variables ($c_1(x_1, x_2) \vee c_2(x_3, x_4)$ does not belong to $\mathcal{CL}$), whereas conjunction of arbitrary constraints is allowed ($c_1(x_1, x_2) \wedge c_2(x_3, x_4)$ belongs to $\mathcal{CL}$).

A *constraint language* is given by a set of relation symbols *Rel* (possibly infinite), a domain U and an *interpretation* function $\mathcal{I} : Rel \to \wp(U \times U)$ which maps relation symbols to binary relations. We assume the function $\mathcal{I}$ to be injective, i.e. different relation symbols are mapped to different relations. The set of variables occurring in a constraint C is denoted by $vars(C)$. A constraint C is evaluated to true or false with respect to the interpretation $\mathcal{I}$ and a variable assignment $\delta : vars(C) \to U$ by a predicate $[\![\cdot]\!]^{\mathcal{I}}_{\delta}$ defined inductively as follows:

$$\begin{aligned}
[\![r(x_1, x_2)]\!]^{\mathcal{I}}_{\delta} = \mathsf{true} &\quad \text{iff} \quad (\delta(x_1), \delta(x_2)) \in r^{\mathcal{I}}, \\
[\![c_1(x_1, x_2) \vee c_2(x_1, x_2)]\!]^{\mathcal{I}}_{\delta} = \mathsf{true} &\quad \text{iff} \quad [\![c_1(x_1, x_2)]\!]^{\mathcal{I}}_{\delta} = \mathsf{true} \text{ or } [\![c_2(x_1, x_2)]\!]^{\mathcal{I}}_{\delta} = \mathsf{true}, \\
[\![\neg c(x_1, x_2)]\!]^{\mathcal{I}}_{\delta} = \mathsf{true} &\quad \text{iff} \quad [\![c(x_1, x_2)]\!]^{\mathcal{I}}_{\delta} = \mathsf{false}, \\
[\![C_1 \wedge C_2]\!]^{\mathcal{I}}_{\delta} = \mathsf{true} &\quad \text{iff} \quad [\![C_1]\!]^{\mathcal{I}}_{\delta} = \mathsf{true} \text{ and } [\![C_2]\!]^{\mathcal{I}}_{\delta} = \mathsf{true}.
\end{aligned}$$

An empty constraint is always evaluated to true. A constraint C is said to be *satisfiable* if there exists a variable assignment δ such that $[\![C]\!]^{\mathcal{I}}_{\delta} = \mathsf{true}$. We say that a constraint C_1 *entails* a constraint C_2, in symbols $C_1 \models_{\mathcal{I}} C_2$, if from $[\![C_1]\!]^{\mathcal{I}}_{\delta} = \mathsf{true}$ it follows that $[\![C_2]\!]^{\mathcal{I}}_{\delta} = \mathsf{true}$ for any variable assignment $\delta : vars(C_1) \cup vars(C_2) \to U$. Constraints C_1 and C_2 are said to be *equivalent*, in symbols $C_1 \equiv_{\mathcal{I}} C_2$, if $C_1 \models_{\mathcal{I}} C_2$ and $C_2 \models_{\mathcal{I}} C_1$.

4.2. Qualitative calculus of a constraint language

A qualitative calculus of a constraint language is a set of rules for transforming one constraint into another in a way that its satisfiability is preserved, i.e. the resulting constraint is entailed from the initial one.

Definition 2 (Relation-type qualitative calculus). A (relation-type) *qualitative calculus* is a pair $(\mathcal{A}, \mathcal{I})$, where $\mathcal{A} = (Rel, +, \cdot, ^-, 0, 1, ;, ^\smile, 1')$ is a relation-type algebra with an underlying set of relation symbols *Rel*, and $\mathcal{I}$ is an interpretation of *Rel* over some domain U, such that

$$r(x, y) \vee s(x, y) \equiv_{\mathcal{I}} (r + s)(x, y) \tag{IR1}$$

$$r(x, y) \wedge s(x, y) \equiv_{\mathcal{I}} (r \cdot s)(x, y) \tag{IR2}$$

$$\neg r(x, y) \equiv_{\mathcal{I}} \overline{r}(x, y) \tag{IR3}$$

$$\models_{\mathcal{I}} 1(x,y) \tag{IR4}$$

$$\models_{\mathcal{I}} \neg 0(x,y) \tag{IR5}$$

$$r(x,y) \wedge s(y,z) \models_{\mathcal{I}} (r;s)(x,z) \tag{IR6}$$

$$r(x,y) \models_{\mathcal{I}} r^{\smile}(y,x) \tag{IR7}$$

$$\models_{\mathcal{I}} 1'(x,x) \tag{IR8}$$

for all $r, s \in Rel$. $\mathcal{A}$ is called a *qualitative algebra* of $\mathcal{I}$, and $\mathcal{I}$ an interpretation of $\mathcal{A}$.

One can think of qualitative calculi as a binary relation between relation-type algebras and constraint languages, given by IR1–IR8.

If $(\mathcal{A}, \mathcal{I})$ is a qualitative calculus, then each constraint C can be transformed into an equivalent constraint in the form $\wedge_{x,y \in vars(C)} r_{xy}(x,y)$, using the rules IR1, IR2, IR3 and IR4. Such a constraint is usually expressed as a directed labeled graph $(vars(C), \lambda)$ called an $\mathcal{A}$-*network* (Hirsch, 1997), where $\lambda : vars(C) \times vars(C) \to Rel$ and $\lambda(x,y) = r_{xy}$. A network (X, λ) is said to be *algebraically closed*, if

$$\lambda(x,x) \leq 1' \qquad \lambda(x,z) \leq \lambda(x,y); \lambda(y,z) \qquad \lambda(x,y) \leq \lambda(y,x)^{\smile} \tag{7}$$

for all $x, y, z \in X$. There is a procedure for refining a network into an algebraically closed one using the rules IR2, IR6, IR7, IR8. If Rel is finite, then this procedure is guaranteed to terminate. An algebraically closed network is said to be *consistent* if $\lambda(x,y) \neq 0$ for all $x, y \in X$, otherwise *inconsistent*. By IR5, an inconsistent network is unsatisfiable.

4.3. Minimality condition

If $\mathcal{A}$ and $\mathcal{B}$ are qualitative algebras for $\mathcal{I}$, then we say that $\mathcal{A}$ is stronger than $\mathcal{B}$, noted as $\mathcal{A} \preceq_{\mathcal{I}} \mathcal{B}$, if

$$(r ;^{\mathcal{A}} s)^{\mathcal{I}} \subseteq (r ;^{\mathcal{B}} s)^{\mathcal{I}} \qquad (r^{\smile \mathcal{A}})^{\mathcal{I}} \subseteq (r^{\smile \mathcal{B}})^{\mathcal{I}} \qquad (1'^{\mathcal{A}})^{\mathcal{I}} \subseteq (1'^{\mathcal{B}})^{\mathcal{I}} \tag{8}$$

for all $r, s \in Rel$. If $\mathcal{A}$ is stronger than all qualitative algebras for a given constraint language $\mathcal{I}$, then $(\mathcal{A}, \mathcal{I})$ is called a *proper qualitative calculus*. $(\mathcal{A}, \mathcal{I})$ is a proper qualitative calculus if and only if $\mathcal{A}$ is the least element with respect to the partial order $\preceq_{\mathcal{I}}$. This minimality condition ensures that $\mathcal{A}$ provides the best calculus for $\mathcal{I}$, w.r.t. the inference rules template IR1–IR8. As will be seen later, not all constraint languages have a proper qualitative calculus.

4.4. General case

The definition of a qualitative calculus given above depends on the following parameters: the constraint logic, the algebraic signature and the set of inference rules. This is the reason why we call it contextual. These parameters can be changed, yielding a different type (or different framework) of qualitative calculi.

It is also important to mention that one can define binary qualitative algebras of non-relation type that still "support" constraint propagation. For example, constraint algebras in the sense of (Nebel and Scivos, 2002) are also qualitative calculi, but of a different type. First, their constraint logic does not have disjunction or negation operators, but has the equality predicate:

$$C ::= r(x_1, x_2) \mid x_1 = x_2 \mid C_1 \wedge C_2, \tag{9}$$

The algebraic signature is $\cdot, 1, ;, ^{\smile}, 1'$, and inference rules are IR2, IR4, together with

$$r(x,y) \wedge s(y,z) \equiv_{\mathcal{I}} (r;s)(x,z) \qquad r(x,y) \equiv_{\mathcal{I}} r^{\smile}(y,x) \qquad x = y \equiv_{\mathcal{I}} 1'(x,y)$$

Qualitative calculi based on inf-semilattices discussed in (Düntsch, 2005) also belong to this framework.

In this paper, we confine ourselves to proper relation-type qualitative calculi.

5. Relation-type qualitative algebras

In the previous section we have set up a framework for relation-type qualitative calculi by fixing a constraint logic, an algebraic signature and a set of inference rules. In this section, we describe the class of qualitative algebras of proper qualitative calculi "internally", that is, in a constructive way. We give two definitions of relation-type qualitative algebras: intensional and extensional, and prove their equivalence in the universal-algebraic sense.

5.1. Extensionally defined qualitative algebras

A constraint language $(Rel, U, \mathcal{I})$ has a proper qualitative calculus only if $Rel^{\mathcal{I}}$ is closed under set union and complement $(-_{U\times U})$, as follows from IR1 and IR3. Assume $\mathcal{B}$ is an arbitrary field of binary relations with a square unit, i.e. with a unit of the form $U\times U$. In other words, $\mathcal{B}$ is a subalgebra of $\wp(U\times U)$. We shall expand $\mathcal{B}$ with operations $\diamond$, $\breve{}$ and a constant $1'$ in a way that the resulting algebra is isomorphic to some proper qualitative algebra. Reformulating IR6–IR8 and the minimality condition 8, we obtain that $\diamond$, $\breve{}$ and $1'$ must satisfy the following conditions.

- For any $R, S \in \mathcal{B}$ and any $x, y, z \in U$, if $(x, y) \in R$ and $(y, z) \in S$, then $(x, z) \in R \diamond S$. Moreover, $R \diamond S$ is the least $\mathcal{B}$-relation that satisfies the above condition for R and S.
- For any $R \in \mathcal{B}$ and any $x, y \in U$, if $(x, y) \in R$, then $(y, x) \in R^{\smile}$. Moreover, $R^{\smile}$ is the least $\mathcal{B}$-relation that satisfies the above condition for R.
- $1'$ is the least $\mathcal{B}$-relation that contains (x, x) for all $x \in U$.

Here is the formal definition.

Definition 3 (Inferential operations). Let $\mathcal{B} = (B, \cup, \cap, -_{U\times U}, \varnothing, U\times U)$ be a field of binary relations.

- $\diamond : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ is called *inferential composition on* $\mathcal{B}$ if

$$R \diamond S = \min\{T \in \mathcal{B} \mid \forall x, y, z.\ R(x, y) \wedge S(y, z) \to T(x, z)\}, \tag{OP1}$$

  for all $R, S \in \mathcal{B}$.

- $\breve{} : \mathcal{B} \to \mathcal{B}$ is called *inferential converse on* $\mathcal{B}$ if

$$R^{\smile} = \min\{T \in \mathcal{B} \mid \forall x, y.\ R(x, y) \to T(y, x)\}, \tag{OP2}$$

  for all $R \in \mathcal{B}$.

- $1'_{\mathcal{B}} \in \mathcal{B}$ is called *inferential identity on* $\mathcal{B}$ if

$$1'_{\mathcal{B}} = \min\{T \in \mathcal{B} \mid \forall x.\ T(x, x)\}. \tag{OP3}$$

The minima in OP1–OP3 exist if and only if the infima of the respective sets, taken in $\wp(U\times U)$, belong to $\mathcal{B}$. Let $\mathcal{F}(X)$ denote the set of all elements of $\mathcal{B}$ greater than X:

$$\mathcal{F}(X) = \{Y \in \mathcal{B} \mid Y \supseteq X\}. \tag{10}$$

One can give an equivalent definition of inferential relational operations that resorts to the conventional composition, converse and identity:

$$R \diamond S = \cap\mathcal{F}(R \circ S), \qquad R^{\smile} = \cap\mathcal{F}(R^{-1}), \qquad 1'_{\mathcal{B}} = \cap\mathcal{F}(Id_U). \tag{11}$$

The operations of inferential composition $\diamond$ and inferential converse $\breve{}$ are known in the literature as *weak composition* and *weak converse*. We prefer to call them inferential, since these operations ensure validity of inference rules IR6 and IR7 respectively. If an inferential composition (converse, identity) corresponds to the conventional one, then it is said to be *strong*, otherwise *weak*.

We name the necessary and sufficient condition on $\mathcal{B}$ that allows to define inferential composition, converse and identity as *inference completeness*.

Definition 4 (Inference completeness). A subalgebra $\mathcal{B}$ of $\wp(U \times U)$ is said to be *inference complete*, if, for all $R, S \in \mathcal{B}$, the sets $\mathcal{F}(R \circ S)$, $\mathcal{F}(R^{-1})$ and $\mathcal{F}(Id_U)$ have least elements, or equivalently,

$$\cap\mathcal{F}(R \circ S) \in \mathcal{B}, \qquad \cap\mathcal{F}(R^{-1}) \in \mathcal{B}, \qquad \cap\mathcal{F}(Id_U) \in \mathcal{B}. \tag{12}$$

An obviously sufficient condition for inference completeness is when $\mathcal{B}$ is closed under composition and converse, and contains the identity relation.

The expansion of $\mathcal{B}$ with the inferential relational operations $\diamond, \breve{}$ and $1'$ is called a *concrete qualitative algebra*.

Definition 5 (Concrete qualitative algebra). Let $\mathcal{B} = (B, \cup, \cap, -_{U\times U}, \varnothing, U\times U)$ be an inference complete field of binary relations. Then the algebra

$$\mathcal{Q}(\mathcal{B}) = (B, \cup, \cap, -_{U\times U}, \varnothing, U\times U, \diamond, \breve{}, 1'_{\mathcal{B}}) \tag{13}$$

is called the *concrete qualitative algebra* generated by $\mathcal{B}$.

The next proposition shows that the operations $\diamond$ and $\breve{}$ are additive, hence concrete qualitative algebras are Boolean algebras with operators. Moreover, $\diamond$ and $\breve{}$ completely distribute over arbitrary unions that belong to $\mathcal{B}$.

Proposition 1. *Let $\mathcal{B}$ be an inference complete subalgebra of $\wp(U \times U)$. If $R_i, S \in \mathcal{B}$ for all i from an arbitrary index set I, and $\cup_{i\in I} R_i \in \mathcal{B}$, then*

$$(\cup_{i\in I} R_i) \diamond S = \cup_{i\in I}(R_i \diamond S) \tag{14}$$

$$S \diamond (\cup_{i\in I} R_i) = \cup_{i\in I}(S \diamond R_i) \tag{15}$$

$$(\cup_{i\in I} R_i)\breve{} = \cup_{i\in I}(R_i\breve{}) \tag{16}$$

Proof. The proof is given for 14 only, 15 and 16 are proven similarly. By definition of $\diamond$, $(\cup_{i\in I} R_i) \diamond S = \min \mathcal{F}((\cup_{i\in I} R_i) \circ T)$. Due to complete distributivity of $\circ$ over $\cup$, $\mathcal{F}((\cup_{i\in I} R_i) \circ S) = \mathcal{F}(\cup_{i\in I}(R_i \circ S))$. Observe that $\mathcal{F}(\cup_{i\in I}(R_i \circ S)) = \cap_{i\in I}\mathcal{F}(R_i \circ S)$. Indeed, for any $T \in \mathcal{B}$, $T \supseteq \cup_{i\in I}(R_i \circ S)$ iff $T \supseteq R_i \circ S$ for all $i \in I$. As a result,

$$(\cup_{i\in I} R_i) \diamond S = \min(\cap_{i\in I}\mathcal{F}(R_i \circ S)). \tag{17}$$

Since $\mathcal{F}(\cup_{i\in I}(R_i \circ S)) \subseteq \mathcal{F}(R_i \circ S)$ for all $i \in I$, we have that $\min \mathcal{F}(R_i \circ S) \subseteq \min \mathcal{F}(\cup_{i\in I}(R_i \circ S))$ for all $i \in I$, therefore

$$\cup_{i\in I} \min \mathcal{F}(R_i \circ S) \subseteq \min \mathcal{F}(\cup_{i\in I}(R_i \circ S)). \tag{18}$$

On the other hand, $\cup_{i\in I} \min \mathcal{F}(R_i \circ S) \in \mathcal{F}(R_i \circ S)$ for all $i \in I$, hence $\cup_{i\in I} \min \mathcal{F}(R_i \circ S) \in \cup_{i\in I}\mathcal{F}(R_i \circ S)$. Consequently,

$$\cup_{i\in I} \min \mathcal{F}(R_i \circ S) \supseteq \min(\cap_{i\in I}\mathcal{F}(R_i \circ S)). \tag{19}$$

From 18 and 19 we obtain

$$\cup_{i\in I} \min \mathcal{F}(R_i \circ S) = \min(\cap_{i\in I}\mathcal{F}(R_i \circ S)). \tag{20}$$

Since $R_i \diamond S = \min \mathcal{F}(R_i \circ S)$ and due to 17 we finally obtain $(\cup_{i\in I} R_i) \diamond S = \cup_{i\in I}(R_i \diamond S)$. □

A concrete qualitative algebra $\mathcal{A}$ is said to be atomic, if so is its Boolean reduct. The next proposition shows that in atomic algebras the inferential operations have yet another equivalent definition.

Proposition 2. *If $\mathcal{A}$ is an atomic concrete qualitative algebra, then*

$$R \diamond S = \sup\{T \in At(\mathcal{A}) \mid (R \circ S) \cap T \neq \varnothing\} \tag{21}$$

$$= \sup\{T \in At(\mathcal{A}) \mid \exists x, y, z.\ R(x, y) \wedge S(y, z) \wedge T(x, z)\} \tag{22}$$

$$R\breve{} = \sup\{T \in At(\mathcal{A}) \mid R^{-1} \cap T \neq \varnothing\} \tag{23}$$

$$= \sup\{T \in At(\mathcal{A}) \mid \exists x, y.\ R(x, y) \wedge T(y, x)\} \tag{24}$$

$$1' = \sup\{T \in At(\mathcal{A}) \mid T \cap Id_U \neq \varnothing\} \tag{25}$$
$$= \sup\{T \in At(\mathcal{A}) \mid \exists x.\ T(x,x)\} \tag{26}$$

for all $R, S \in \mathcal{A}$.

Proof. Let T be an arbitrary atom of $\mathcal{A}$. Assume that $(R \circ S) \cap T \neq \varnothing$. Since $R \diamond S \supseteq R \circ S$, it follows that $(R \diamond S) \cap T \neq \varnothing$. As T is an atom, we conclude that $T \subseteq R \diamond S$. Conversely, assume $T \subseteq R \diamond S$. If $(R \circ S) \cap T = \varnothing$, then $(R \diamond S) \backslash T \in \mathcal{A}$ and $(R \diamond S) \backslash T \supseteq R \circ S$, thus $(R \diamond S) \backslash T \in \mathcal{F}(R \circ S)$ and $(R \diamond S) \backslash T \subset R \diamond S$, which is a contradiction. Thus, $(R \circ S) \cap T \neq \varnothing$. We have shown that $At(R \diamond S) = \{T \in At(\mathcal{A}) \mid (R \circ S) \cap T \neq \varnothing\}$. By definition of atomicity, $R \diamond S = \sup At(R \diamond S)$. Equation 22 is obtained from 21 by expressing the relation $(R \circ S) \cap T \neq \varnothing$ axiomatically. Equations 23–26 are proven similarly. □

$\mathfrak{B}$ denotes the class of all inference complete fields of binary relations with a square unit. By QA, or $\mathsf{QA}(\mathfrak{B})$, we denote the class of all relation-type algebras isomorphic to some concrete qualitative algebra. The algebras that belong to QA are called *qualitative algebras.* A class of algebras is called a variety if it is closed under homomorphic images, subalgebras and direct products. The next proposition shows that qualitative algebras are not directly decomposable, hence do not form a variety.

Proposition 3. *If $\mathcal{A} \in \mathsf{QA}$, then it is directly indecomposable.*

Proof. From $\mathcal{A} \in \mathsf{QA}$ it follows that there is a $\mathcal{B} \in \mathfrak{B}$ such that $\mathcal{A} \cong \mathcal{Q}(\mathcal{B})$. For any $R \in \mathcal{B}$ such that $R \neq \varnothing$,

$$(1 \diamond R) \diamond 1 \supseteq (1 \circ R) \circ 1 = 1. \tag{27}$$

Assume that $\mathcal{B}$ is directly decomposable, i.e. there exist inference complete fields of binary relations $\mathcal{B}', \mathcal{B}'' \in \mathfrak{B}$ such that $\mathcal{Q}(\mathcal{B}) \cong \mathcal{Q}(\mathcal{B}') \otimes \mathcal{Q}(\mathcal{B}'')$. Let $\mathcal{A}' = \mathcal{Q}(\mathcal{B}') \otimes \mathcal{Q}(\mathcal{B}'')$. Then, for any $R \in \mathcal{Q}(\mathcal{B}')$, $(R, \varnothing) \in \mathcal{A}'$. However,

$$\big((1,1);(R,\varnothing)\big)\,;(1,1) = \big((1 \diamond R) \diamond 1, \varnothing\big) \neq (1,1), \tag{28}$$

which contradicts (27). □

5.2. Intensionally defined qualitative algebras

Qualitative algebras, unlike well-known classes NA, WA, SA and RA, are not an algebraic variety, as shown in Proposition 3, thus cannot be defined by means of equational axioms. However, there exists a different intensional way of defining the class of qualitative algebras, based on a certain representability property.

Definition 6 (Q-homomorphism). Let $\mathcal{A} = (A, +, \cdot, ^-, 0_\mathcal{A}, 1_\mathcal{A}, ;, ^\smile, 1'_\mathcal{A})$ and $\mathcal{B} = (B, +, \cdot, ^-, 0_\mathcal{B}, 1_\mathcal{B}, ;, ^\smile, 1'_\mathcal{B})$ be Boolean algebras with operators. An injective function $\varphi : A \to B$ is called a *Q-homomorphism* if

(Q1) $\varphi(a + b) = \varphi(a) + \varphi(b)$

(Q2) $\varphi(\overline{a}) = \overline{\varphi(a)}$

(Q3) $a; b \leq c \leftrightarrow \varphi(a); \varphi(b) \leq \varphi(c)$

(Q4) $a^\smile \leq b \leftrightarrow \varphi(a)^\smile \leq \varphi(b)$

(Q5) $1'_\mathcal{A} \leq a \leftrightarrow 1'_\mathcal{B} \leq \varphi(a)$

for all $a, b, c \in A$.

Q-homomorphisms are Boolean algebra homomorphisms, but not necessarily BAO homomorphisms, since they do not necessarily preserve ;, $^\smile$ and $1'$. However, any injective BAO homomorphism is a Q-homomorphism. Composition of Q-homomorphisms is a Q-homomorphism. As a consequence, relation-type BAOs together with Q-homomorphisms form a category.

A Q-homomorphism to an algebra $\mathfrak{Re}(U)$ of binary relations on some set U is called a *Q-representation.* A BAO is said to be *Q-representable* if it has a Q-representation. An interesting observation is that composition of a Q-homomorphism with a Q-representation is a Q-representation. Moreover, Q-representable algebras are closed under Q-homomorphic images. Thus, if $\mathcal{A}'$ is Q-representable and $\varphi : \mathcal{A} \to \mathcal{A}'$ is a Q-homomorphism, then $\mathcal{A}$ is Q-representable. The class of Q-representable relation-type Boolean algebras with operators is denoted by QR, or $\mathsf{QR}(\mathsf{BAO})$. If $\mathfrak{A}$ is a subclass of BAO, then $\mathsf{QR}(\mathfrak{A})$ will denote the class of qualitatively representable $\mathfrak{A}$.

Theorem 1. *If $\mathcal{A}$ is a relation-type algebra and $\varphi : \mathcal{A} \to \mathfrak{Re}(U)$ is a Q-representation, then $\varphi(\mathcal{A}) \in \mathfrak{B}$ and $\varphi : \mathcal{A} \to \mathcal{Q}(\varphi(\mathcal{A}))$ is an isomorphism of qualitative algebras. Conversely, if $\mathcal{B}$ is an inference complete square field of binary relations over a set U and $\varphi : \mathcal{A} \to \mathcal{Q}(\mathcal{B})$ is an isomorphism of qualitative algebras, then $\varphi : \mathcal{A} \to \mathfrak{Re}(U)$ is a Q-representation.*

Proof. Observe that $(\varphi(\mathcal{A}), \cup, \cap, -_{U\times U}, \varnothing, U \times U)$ is a field of binary relations isomorphic to the Boolean reduct of $\mathcal{A}$. Let us prove that for all $r, s \in \mathcal{A}$, $\varphi(r) \diamond \varphi(s)$ is well-defined and equal to $\varphi(r; s)$. Since $r; s \geq r; s$, from the definition of Q-representation it follows that $\varphi(r; s) \supseteq \varphi(r) \circ \varphi(s)$, hence $\varphi(r; s) \in \mathcal{F}(\varphi(r) \circ \varphi(s))$. Let $\varphi(t)$ be an arbitrary element of $\varphi(\mathcal{A})$. If $\varphi(t) \supseteq \varphi(r) \circ \varphi(s)$, then $t \geq r; s$, therefore, since φ is order-preserving, $\varphi(t) \supseteq \varphi(r; s)$, which means that $\varphi(r; s) = \min \mathcal{F}(\varphi(r) \circ \varphi(s)) = \varphi(r) \diamond \varphi(s)$. Similarly, we show that $\varphi(r^{\smallsmile}) = \min \mathcal{F}(\varphi(r)^{-1})$ for all $r \in \mathcal{A}$ and $\varphi(1'_{\mathcal{A}}) = \min \mathcal{F}(Id_U)$, which implies that $\varphi(r^{\smallsmile}) = \varphi(r)^{\smallsmile}$ and $\varphi(1'_{\mathcal{A}}) = 1'_{\varphi(\mathcal{A})}$. Thus, φ is an isomorphism between $\mathcal{A}$ and $\mathcal{Q}(\varphi(\mathcal{A}))$.

Conversely, assume that $\mathcal{B}$ is some inference complete field of binary relations over a set U and $\varphi : \mathcal{A} \to \mathcal{Q}(\mathcal{B})$ is an isomorphism. For any $a, b, c \in \mathcal{A}$, $c \geq a; b$ iff $\varphi(c) \supseteq \varphi(a) \diamond \varphi(b)$ iff $\varphi(c) \supseteq \varphi(a) \circ \varphi(b)$. Likewise, $c \geq a^{\smallsmile}$ iff $\varphi(c) \supseteq \varphi(a)^{-1}$ and $c \geq 1'_{\mathcal{A}}$ iff $\varphi(c) \supseteq Id_U$. Therefore, φ is a Q-representation of $\mathcal{A}$ over U, hence $\mathcal{A} \in \mathsf{QR}$. □

Corollary 1. $\mathsf{QR} = \mathsf{QA}$.

Proof. Vacuous. □

Corollary 2. *If $\mathfrak{A}$ is a subclass of* BAO*, then* $\mathsf{QA} \cap \mathfrak{A} = \mathsf{QR}(\mathfrak{A})$.

Proof. Follows from $\mathsf{QR}(\mathfrak{A}) = \mathsf{QR} \cap \mathfrak{A}$ and $\mathsf{QR} = \mathsf{QA}$. □

Corollary 2 indicates that Q-representability is a way of describing the intersection of qualitative algebras with other classes of algebras, such as non-associative relation algebras. This will be used in the next section.

Finally, Proposition 4 shows that a Q-homomorphism from an atomic BAO can be defined alternatively, in a way that generalizes semi-strong representations (Mossakowski et al., 2006).

Proposition 4. *Let $\mathcal{A} = (A, +, \cdot, ^-, 0_{\mathcal{A}}, 1_{\mathcal{A}}, ;, ^{\smallsmile}, 1'_{\mathcal{A}})$ and $\mathcal{B} = (B, +, \cdot, ^-, 0_{\mathcal{B}}, 1_{\mathcal{B}}, ;, ^{\smallsmile}, 1'_{\mathcal{B}})$ be BAOs and assume that $\mathcal{A}$ is atomic. An injective function $\varphi : A \to B$ is a Q-homomorphism iff*

(S1) $\varphi(a + b) = \varphi(a) + \varphi(b)$

(S2) $\varphi(\overline{a}) = \overline{\varphi(a)}$

(S3) $\varphi(a; b) \geq \varphi(a); \varphi(b)$

(S4) $\varphi(a^{\smallsmile}) \geq \varphi(a)^{\smallsmile}$

(S5) $\varphi(1'_{\mathcal{A}}) \geq 1'_{\mathcal{B}}$

(S6) $At(a; b) = \{c \in At(\mathcal{A}) \mid \varphi(c) \cdot (\varphi(a); \varphi(b)) \neq 0_{\mathcal{B}}\}$

(S7) $At(a^{\smallsmile}) = \{c \in At(\mathcal{A}) \mid \varphi(c) \cdot \varphi(a)^{\smallsmile} \neq 0_{\mathcal{B}}\}$

(S8) $At(1'_{\mathcal{A}}) = \{c \in At(\mathcal{A}) \mid \varphi(c) \cdot 1'_{\mathcal{B}} \neq 0_{\mathcal{B}}\}$

for all $a, b, c \in A$.

Proof. Let φ be a Q-homomorphism and $a, b \in A$. From $a; b \geq a; b$ and Q3 it follows that $\varphi(a; b) \geq \varphi(a); \varphi(b)$. Similarly we obtain S4 and S5. To prove S6, let $c \in At(a; b)$ and assume $\varphi(c) \cdot (\varphi(a); \varphi(b)) = 0_{\mathcal{B}}$. This implies that $(\varphi(a); \varphi(b)) \cdot \overline{\varphi(c)} = \varphi(a); \varphi(b)$. Further, $\varphi((a; b) \cdot \bar{c}) = \varphi(a; b) \cdot \varphi(\bar{c}) = \varphi(a; b) \cdot \overline{\varphi(c)} \geq (\varphi(a); \varphi(b)) \cdot \overline{\varphi(c)} = \varphi(a); \varphi(b)$, therefore $(a; b) \cdot \bar{c} \geq a; b$, which contradicts $c \in At(a; b)$. Thus, $\varphi(c) \cdot (\varphi(a); \varphi(b)) \neq 0_{\mathcal{B}}$. Conversely, assume $c \in At(\mathcal{A})$ and $\varphi(c) \cdot (\varphi(a); \varphi(b)) \neq 0_{\mathcal{B}}$. Due to $\varphi(a; b) \geq \varphi(a); \varphi(b)$ we have $\varphi(c) \cdot \varphi(a; b) \neq 0_{\mathcal{B}}$, therefore $c \cdot (a; b) \neq 0_{\mathcal{A}}$, which means that $c \in At(a; b)$. S7 and S8 are proven similarly.

Assume now that S1–S8 hold and let us prove Q3. If $a, b, c \in A$ and $c \geq a; b$, then $\varphi(c) \geq \varphi(a; b) \geq \varphi(a); \varphi(b)$. Conversely, assume that $\varphi(c) \geq \varphi(a); \varphi(b)$. If $d \in At(\mathcal{A})$ and $d \leq a; b$, then $\varphi(d) \cdot (\varphi(a); \varphi(b)) \neq 0_{\mathcal{B}}$, hence $\varphi(d) \cdot \varphi(c) \neq 0_{\mathcal{B}}$, which implies $d \leq c$. Thus, $At(a; b) \subseteq At(c)$, therefore $a; b \leq c$. Q4 and Q5 are proven similarly. □

6. Important subclasses of relation-type qualitative algebras

In the previous section, we have defined the broadest class of relation-type qualitative algebras in two equivalent ways: as Q-representable BAOs and as isomorphism closure of algebras generated by inference complete fields of binary relations. This gives us two ways of describing subclasses of qualitative algebras: one intensional, or algebraic, and the other extensional, or semantic. In this section, we will consider two important subclasses of qualitative algebras: those that stem from JEPD relations and those that are non-associative relation algebras, as well as their intersection.

6.1. Qualitative algebras of JEPD relations

Inference completeness of a Boolean set algebra $\mathcal{B}$ with unit element $U \times U$ is not easy to check in practice. Completeness of $\mathcal{B}$ may not ensure its inference completeness. But if $\mathcal{B}$ is both complete and atomic, then it is also inference complete. A complete atomic $\mathcal{B}$ is fully specified by the set of its atoms $At(\mathcal{B})$, since $\mathcal{B}$ can be obtained from $At(\mathcal{B})$ by taking its union closure. The set $At(\mathcal{B})$ consists of nonempty binary relations on U, which are jointly exhaustive and pairwise disjoint (JEPD), meaning that each $(x, y) \in U \times U$ belongs to one and only one relation from $At(\mathcal{B})$. In the sequel, we will refer to any set of nonempty JEPD relations as a *partition scheme*. If $\mathcal{P}$ is a partition scheme, then by $\mathcal{P}_\cup$ we will denote its union closure. $\mathcal{P}_\cup$ is a field of binary relations. It is easy to check that $\mathcal{P}_\cup$ is inference complete.

Proposition 5. *If $\mathcal{P}$ is a partition scheme, then the operations $\diamond$ and $\breve{}$ of $\mathcal{Q}(\mathcal{P}_\cup)$ are completely additive.*

Proof. Follows from Proposition 1 and the fact that the field of binary relations $\mathcal{P}_\cup$ is closed under arbitrary unions. □

It follows from Proposition 5 that qualitative algebras generated by partition schemes are fully specified by their set of atoms, the subset of sub-identity atoms, and operations ; and $\breve{}$ restricted to atoms. This is called the *atom structure* of a qualitative algebra.

Let $\mathfrak{P}$ denote the class of all finite partition schemes. Then $\mathsf{QA}(\mathfrak{P})$ denotes the class of qualitative algebras that stem from finite partition schemes, i.e. are isomorphic to $\mathcal{Q}(\mathcal{P}_\cup)$ for some $\mathcal{P} \in \mathfrak{P}$. From Proposition 5 it follows that $\mathsf{QA}(\mathfrak{P}) = \mathsf{QA} \cap \mathsf{FBAO}$, where FBAO is the class of finite BAOs. Applying Corollary 2 we obtain the following:

Proposition 6. *Qualitative algebras generated by finite partition schemes are finite Q-representable complete atomic BAOs, in symbols* $\mathsf{QA}(\mathfrak{P}) = \mathsf{QR}(\mathsf{FBAO})$.

Proof. The inclusion $\mathsf{QA}(\mathfrak{P}) \subseteq \mathsf{QR}(\mathsf{FBAO})$ is obvious. Conversely, if $\mathcal{A}$ is a finite BAO and $\varphi : \mathcal{A} \to \mathfrak{Re}(U)$ is a Q-representation, then $\mathcal{P} := \varphi(At(\mathcal{A}))$ is a finite partition scheme over U, and $\mathcal{A} \cong \mathcal{Q}(\mathcal{P}_\cup)$, hence $\mathcal{A} \in \mathsf{QA}(\mathfrak{P})$. □

Proposition 6 characterizes qualitative algebras stemming from JEPD relations both intensionally and extensionally. The class QAJEPD (cf. Section 3.2) is the reduct of $\mathsf{QA}(\mathfrak{P})$ which "forgets" the distinguished element $1'$.

6.2. Q-representable non-associative relation algebras

The next theorem establishes that "Q-representation" is essentially a conservative extension of "qualitative representation" (cf. Definition 1) beyond non-associative relation algebras.

Theorem 2. *If $\mathcal{A} \in \mathsf{NA}$ and $\varphi : \mathcal{A} \to \mathfrak{Re}(U)$ is a Q-representation, then $\varphi(1')$ is an equivalence relation on U and $\varphi' : \mathcal{A} \to \mathfrak{Re}(U')$ defined below is a qualitative representation:*

- $U' = {}^{U}/_{\varphi(1')}$,
- $\varphi'(a) = \{(X, Y) \in U' \times U' \mid \exists x \in X, y \in Y.\ (x, y) \in \varphi(a)\}$ *for all* $a \in \mathcal{A}$.

We precede the proof of Theorem 2 with some auxiliary notation and two lemmas. Given a binary relation R over U, by $R(x, \_)$ we denote the set of all $y \in U$ such that $(x, y) \in R$. Likewise, $R(\_, x) = \{y \in U \mid (y, x) \in R\}$. Intuitively, if $R(x, \_) = R(y, \_)$ and $R(\_, x) = R(\_, y)$ for some $x, y \in U$, then R does not distinguish between x and y.

Lemma 1. *Let R, S be arbitrary binary relations and E an equivalence relation.*

1. *If $R = E \circ S$, then $R(x, \_) = R(y, \_)$ for all $(x, y) \in E$.*
2. *If $R = S \circ E$, then $R(\_, x) = R(\_, y)$ for all $(x, y) \in E$.*

Proof. We will only prove the first assertion. Assume $(x, y) \in E$ and $(x, z) \in R$ for some arbitrary z. $(x, z) \in E \circ S \Rightarrow \exists u.\ (x, u) \in E \wedge (u, z) \in S \Rightarrow (y, u) \in E \wedge (u, z) \in S \Rightarrow (y, z) \in R$. Since z was arbitrary, it follows that $R(x, \_) = R(y, \_)$. □

Let $\sim$ be an equivalence relation over U. The equivalence class of $x \in U$ is denoted by $[x]$. Let U' be the set of equivalence classes, i.e. $U' = {}^{U}/_{\sim}$. We define a function ${}^{\cdot}/_{\sim}$ between $\wp(U \times U)$ and $\wp(U' \times U')$ as follows:

$$\begin{array}{rcl} {}^{\cdot}/_{\sim} \ : \ \wp(U \times U) & \to & \wp(U' \times U') \\ \Psi & & \Psi \\ R & \mapsto & {}^{R}/_{\sim} = \{([x], [y]) \in U' \times U' \mid (x, y) \in R\} \end{array} \tag{29}$$

It is easy to check that ${}^{\cdot}/_{\sim}$ is order-preserving: if $R \subseteq S$ for any $R, S \subseteq U \times U$, then ${}^{R}/_{\sim} \subseteq {}^{S}/_{\sim}$. Moreover, ${}^{R^{-1}}/_{\sim} = ({}^{R}/_{\sim})^{-1}$.

Lemma 2. *If $\mathcal{B}$ is an inference complete Boolean subalgebra of $\wp(U \times U)$ and for all $R \in \mathcal{B}$ and all $x \sim x'$ the equalities $R(x, \_) = R(x', \_)$ and $R(\_, x) = R(\_, x')$ hold, then ${}^{\cdot}/_{\sim}$ restricted to $\mathcal{B}$ is an isomorphism between $\mathcal{Q}(\mathcal{B})$ and $\mathcal{Q}({}^{\mathcal{B}}/_{\sim})$, where ${}^{\mathcal{B}}/_{\sim} = \{{}^{R}/_{\sim} \mid R \in \mathcal{B}\}$.*

Proof. From the premises it follows that $(x, y) \in R$ iff $([x], [y]) \in {}^{R}/_{\sim}$ for all $R \in \mathcal{B}$. Let us show that ${}^{\cdot}/_{\sim}$ is injective on $\mathcal{B}$. Indeed, for any $R, S \in \mathcal{B}$, assume that ${}^{R}/_{\sim} = {}^{S}/_{\sim}$. Then $(x, y) \in R \Leftrightarrow ([x], [y]) \in {}^{R}/_{\sim} \Leftrightarrow ([x], [y]) \in {}^{S}/_{\sim} \Leftrightarrow (x, y) \in S$, thus $R = S$. The equalities ${}^{(U \times U) \setminus R}/_{\sim} = (U' \times U') \setminus ({}^{R}/_{\sim})$ and ${}^{R \cup S}/_{\sim} = {}^{R}/_{\sim} \cup {}^{S}/_{\sim}$ for all $R, S \in \mathcal{B}$ are checked vacuously. Further, since $\forall x, y, z \in U.\ R(x, y) \wedge S(y, z) \to T(x, z)$ is equivalent to $\forall [x], [y], [z] \in U'.\ ({}^{R}/_{\sim})([x], [y]) \wedge ({}^{S}/_{\sim})([y], [z]) \to ({}^{T}/_{\sim})([x], [z])$ for all $R, S, T \in \mathcal{B}$, it follows that ${}^{\mathcal{F}(R \circ S)}/_{\sim} = \mathcal{F}({}^{R}/_{\sim} \circ {}^{S}/_{\sim})$. Since ${}^{\cdot}/_{\sim}$ is order-preserving, we conclude that $({}^{R}/_{\sim}) \diamond ({}^{S}/_{\sim})$ is well-defined for all $R, S \in \mathcal{B}$ and equal to ${}^{R \diamond S}/_{\sim}$. Likewise, we prove that $({}^{R}/_{\sim})^{\smile} = {}^{R^{\smile}}/_{\sim}$ and $1'_{\mathcal{B}/\sim} = {}^{1'_{\mathcal{B}}}/_{\sim}$. □

Proof of Theorem 2. Let us prove first that $\varphi(a)^{-1} = \varphi(a^{\smile})$ for all $a \in \mathcal{A}$. Since $\mathcal{A} \in \mathsf{NA}$, $a^{\smile\smile} = a$ for any $a \in \mathcal{A}$. From $a^{\smile} \geq a^{\smile}$ and Q4 we obtain $\varphi(a^{\smile}) \supseteq \varphi(a)^{-1}$. On the other hand, from $a \geq a^{\smile\smile}$ we have $\varphi(a) \supseteq \varphi(a^{\smile})^{-1}$, from which we obtain $\varphi(a)^{-1} \supseteq \varphi(a^{\smile})$. As a consequence, $\varphi(a)^{-1} = \varphi(a^{\smile})$.

Now let us show that $\varphi(1')$ is an equivalence relation. From Theorem 1 it follows that $\mathcal{A} \cong \mathcal{Q}(\varphi(\mathcal{A}))$. Since $1' ; 1' = 1'$ in every non-associative relation algebra, we have that $\varphi(1') \diamond \varphi(1') = \varphi(1')$. But since $\varphi(1') = \varphi(1') \diamond \varphi(1') \supseteq \varphi(1') \circ \varphi(1') \supseteq \varphi(1') \circ Id_U = \varphi(1')$, we conclude that $\varphi(1') \circ \varphi(1') = \varphi(1')$, which means that $\varphi(1')$ is a transitive relation. Since, in every NA, $1'^{\smile} = 1'$ and due to $\varphi(a)^{-1} = \varphi(a^{\smile})$ we conclude that $\varphi(1')$ is a symmetric relation. Finally, $\varphi(1')$ is reflexive, because it contains Id_U, hence $\varphi(1')$ is an equivalence relation.

From $\mathcal{A} \in \mathsf{NA}$ and $\mathcal{A} \cong \mathcal{Q}(\varphi(\mathcal{A}))$, we have that $\varphi(a) = \varphi(a) \diamond \varphi(1') = \varphi(1') \diamond \varphi(a)$ for all $a \in \mathcal{A}$. The chain $\varphi(a) = \varphi(a) \diamond \varphi(1') \supseteq \varphi(a) \circ \varphi(1') \supseteq \varphi(a) \circ Id_U = \varphi(a)$ implies that $\varphi(a) \circ \varphi(1') = \varphi(a)$ for all $a \in \mathcal{A}$. Likewise, $\varphi(1') \circ \varphi(a) = \varphi(a)$.

Applying Lemma 1 we obtain that for any $R \in \varphi(\mathcal{A})$ and any $(x, y) \in \varphi(1')$, $R(x, \_) = R(y, \_)$ and $R(\_, x) = R(\_, y)$. By Lemma 2, $\cdot/\varphi(1')$ is an isomorphism between $\mathcal{Q}(\varphi(\mathcal{A}))$ and $\mathcal{Q}(\varphi(\mathcal{A})/\varphi(1'))$.

Finally, it remains to check that $\varphi' : \mathcal{A} \to \mathfrak{Re}(U')$ is a qualitative representation. Observe that $\varphi'(a) = \varphi(a)/\varphi(1')$ for all $a \in \mathcal{A}$. Since φ' is a composition of Q-homomorphisms φ and $\cdot/\varphi(1')$, it is also a Q-homomorphism. In addition, $\varphi'(1') = Id(U')$ and $\varphi'(a^\smile) = \varphi(a^\smile)/\sim = \varphi(a)^{-1}/\sim = (\varphi(a)/\sim)^{-1} = \varphi'(a)^{-1}$, hence φ' is a qualitative representation. □

Corollary 3. $\mathsf{QR}(\mathsf{NA}) = \mathsf{QRA}$ *(cf. Section 3.1).*

Proof. Vacuously follows from Theorem 2. □

We are interested now in the extensional description of $\mathsf{QR}(\mathsf{NA})$. Hirsch et al. (2019) calls semantic structures corresponding to qualitative representations *"herds"*.

Definition 7 (Hirsch et al. (2019))**.** A field of binary relations $\mathcal{B}$ with a unit element $U \times U$ is called a *herd* if it contains Id_U and is closed under converse $(^{-1})$.

Herds are defined in (Hirsch et al., 2019) without the requirement of inference completeness. However, an inference-incomplete herd cannot be an image of a qualitative representation. An example of a herd which is not inference complete is given below.

Example 3. Consider a field of binary relations $\mathcal{H}$ defined as follows.

$$\mathcal{H} = \{R \subseteq \mathbb{Z} \times \mathbb{Z} \mid \left|R \backslash Id_{\mathbb{Z}}\right| < \omega \text{ or } \left|\mathbb{Z} \times \mathbb{Z} \backslash (R \cup Id_{\mathbb{Z}})\right| < \omega\} \tag{30}$$

$\mathcal{H}$ consists of such binary relations R over $\mathbb{Z}$ that either $R \backslash Id_{\mathbb{Z}}$ is finite or $R \cup Id_{\mathbb{Z}}$ is co-finite. $\mathcal{H}$ is closed under converse $(^{-1})$ and contains the identity relation $Id_{\mathbb{Z}}$. However, it is not inference complete. Indeed, if $R, S \in \mathcal{S}$, $R \backslash Id_{\mathbb{Z}}$ is finite and $S \cup Id_{\mathbb{Z}}$ is co-finite, then $R \circ S \notin \mathcal{S}$. Moreover, the set $\mathcal{F}(R \circ S)$ (cf. 10) consists of such T for which $T \cup Id_{\mathbb{Z}}$ is co-finite, thus $\mathcal{F}(R \circ S)$ has no least element.

The class of inference complete herds is denoted by $\mathfrak{H}$.

Proposition 7. *Q-representable non-associative relation algebras are those generated by inference complete herds, in symbols* $\mathsf{QA}(\mathfrak{H}) = \mathsf{QR}(\mathsf{NA})$.

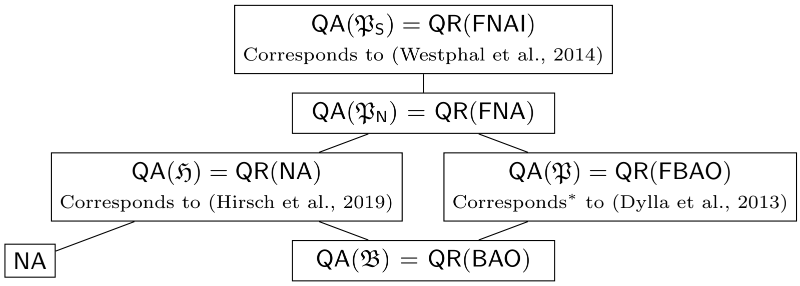
Proof. To prove $\mathsf{QA}(\mathfrak{H}) \subseteq \mathsf{QR}(\mathsf{NA})$ we have to show that concrete qualitative algebras generated by inference complete herds are non-associative relation algebras. The proof is very similar to that of Lemma 2 in (Ligozat and Renz, 2004). The identity law holds due to the equalities $Id_U \diamond R_i = Id_U \circ R_i = R_i = R_i \circ Id_U = R_i \diamond Id_U$. The Peircean law is proven as follows:

$$(R_i \diamond R_j) \cap R_k^{-1} = \varnothing \Leftrightarrow (R_i \circ R_j) \cap R_k^{-1} = \varnothing \Leftrightarrow (R_j \circ R_k) \cap R_i^{-1} = \varnothing \Leftrightarrow (R_j \diamond R_k) \cap R_i^{-1} = \varnothing.$$

If $\mathcal{A} \in \mathsf{NA}$ and $\varphi : \mathcal{A} \to \mathfrak{Re}(U)$ is a Q-representation, then by Theorem 2, φ' is a qualitative representation. Consequently, $\mathcal{H} := \varphi'(\mathcal{A})$ is closed under converse $(^{-1})$ and contains the identity $Id_{U'}$, so $\mathcal{H}$ is a herd. By Theorem 1, φ' is an isomorphism between $\mathcal{A}$ and $\mathcal{Q}(\mathcal{H})$, hence $\mathcal{A} \in \mathsf{QA}(\mathfrak{H})$. □

Let us now describe the intersection of $\mathsf{QA}(\mathfrak{P})$ and $\mathsf{QA}(\mathfrak{H})$. It is generated by a class of partition schemes denoted by $\mathfrak{P}_{\mathsf{N}}$ and defined as follows. A partition scheme $\mathcal{P}$ on a set U belongs to $\mathfrak{P}_{\mathsf{N}}$ iff $\mathcal{P}$ is closed under converse $(^{-1})$ and $Id_U \in \mathcal{P}_{\cup}$. $\mathfrak{P}_{\mathsf{N}}$ is called the class of *non-associative partition schemes* (Inants, 2016). The intensional description of $\mathsf{QA}(\mathfrak{P}_{\mathsf{N}})$ is obtained by intersecting NA with FBAO, which yields the class FNA of finite non-associative Boolean algebras.

Figure 1 summarizes the results of this section. Each of the subclasses of QA mentioned in Figure 1 is closed under Q-homomorphisms, except $\mathsf{QA}(\mathfrak{P}_{\mathsf{S}})$.



* With at least weak composition and converse, and distinguished weak identity.

QA(_)	qualitative algebras generated by _	$\mathfrak{B}$	inference complete fields of binary relations with a square unit
QR(_)	Q-representable _		
BAO	relation-type BAOs	$\mathfrak{H}$	inference complete herds
NA	non-associative relation algebras	$\mathfrak{P}$	finite partition schemes
F_	finite _	$\mathfrak{P}_N$	non-associative partition schemes
_I	_ with atomic identity	$\mathfrak{P}_S$	strong partition schemes

Figure 1: Intersection structure of qualitative algebras' subclasses.

7. Conclusions and future work

We finally come back to Ligozat and Renz (2004)'s question "what is a qualitative calculus?". We reconciled the alternative answers to this question by introducing two functions: QA, which generalizes the extensional definition of Dylla et al. (2013), and QR, which generalizes the intensional definition of Hirsch et al. (2019) (cf. Figure 1). We established the link between the extensional and intensional approaches: a class of semantic structures X generates a class of Q-representable algebras Y if $\mathsf{QA}(X) = \mathsf{QR}(Y)$. We defined the class of relation-type qualitative algebras, which includes both QAJEPD (Dylla et al., 2013) and QRA (Hirsch et al., 2019), and specified the intersection of QAJEPD and QRA. In addition, we have shown that the necessary and sufficient condition for defining a proper relation-type qualitative calculus on a system of binary relations is its closedness under union and complement, and inference completeness. This can be seen as a practical tool to design new calculi.

Along with the two equivalent "internal" definitions of relation-type qualitative calculi, we gave an "external", contextual definition thereof. The contextual definition opens a broader perspective on the paradigm of algebraic constraint-based reasoning and pictures the "relation-type qualitative calculus" framework as one of many possible frameworks for qualitative calculi. A framework for qualitative calculi is determined by a constraint logic, an algebraic signature and a set of inference rules.

The term "qualitative calculus" may be questioned. For example, the calculi $\mathcal{STAR}_m$ (Renz and Mitra, 2004) or $\mathcal{OPRA}_m$ (Moratz, 2006) are studied within the framework of qualitative calculi, but they are arguably quantitative, despite being discrete and finite. Moreover, the definition of Hirsch et al. (2019) admits calculi with infinitely many relations, such as the one in Example 1, and we have no example of an infinite non-quantitative calculus. Indeed, this framework is not bound to the qualitative reasoning field, as it admits quantitative calculi as well. However, we followed the tradition and kept the "qualitative calculus" name, as the overwhelming majority of such calculi already carry this name. Beyond names, it is the contribution of this paper to provide a unified framework to compare these calculi.

This work opens several perspectives for future work. First, with an infinite qualitative algebra we have no guarantee that the algebraic closure procedure would terminate. A way to address it is by introducing a proximity measure between relations and achieving convergence of constraints up to a certain proximity value. Secondly, we plan to expand the framework to n-ary qualitative calculi. The methodology is the same as in this paper: a "calculus" is seen as a certain property of an abstract algebra w.r.t. a constraint language. This property is attributed by the choice of constraint logic, algebraic operations (the signature) and inference rules. Finally, we plan to introduce the subclass of qualitative algebras that allows for defining sorts of objects. For instance, Meiri (1996) and Kurata and Shi (2009) define relations between different kinds of spatial and temporal objects and propose ad-hoc reasoning methods with these heterogeneous relations. The many-sorted framework of qualitative calculi would allow for structurally combining calculi defined for different kinds of objects in a generic way (Inants, 2016). Allowing non-atomic identity elements is the first step in introducing sorts, but it does not guarantee all relations to be well-sorted, which calls for imposing additional constraints on relations.

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